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A RELATIONAL FRAME AND ARTIFICIAL NEURAL NETWORK APPROACH TO COMPUTER-INTERACTIVE MATHEMATICS

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Fifteen participants unfamiliar with mathematical operations relative to reflections and vertical and horizontal shifts were exposed to an introductory lecture regarding the fundamentals of the rectangular coordinate system and the relationship between formulas and their graphed analogues. The lecture was followed immediately by computer-assisted instructions and matching-to-sample procedures in which participants were exposed to computer-posted rules regarding the relationship between particular types of formulas and their respective graphs. After participants demonstrated mutual entailment on formula-to-graph and graph-to-formula functions, they were assessed for 36 novel relations on complex variations of the original training formulas and graphs. In Experiment 1, 5 of 15 participants demonstrated perfect or near perfect performance on all novel relationships. Experiment 2 was directed at the remaining 10 participants who failed to correctly identify all mathematical relationships assessed in Experiment 1. The error patterns for these 10 participants were classified with the help of an artificial neural network self-organizing map (SOM). Training in Experiment 2 was directed exclusively at the types of errors identified by the SOM. Following remedial training, all participants demonstrated a substantial reduction in errors compared to their performance in Experiment 1. Derived transfer of stimulus control using mathematical relations is discussed.

Many everyday occurrences entail two or more elements that are associated by some rule of correspondence. The mathematical term for such a correspondence is a *relation*. Within the stimulus equivalence

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literature, one of the few applied studies to address learning mathematical relations confined its analysis to fractions and their decimal equivalents. Lynch and Cuvo (1995) developed a protocol that provided low performing fifth- and sixth-grade students with an opportunity to match fraction ratios (A) to their graph/pictorial illustrations (B). Subsequently, students were trained to match the pictorials (B) to their corresponding decimal values (C). Following the emergence of equivalence, participants were tested for generalization on other (untrained) fraction-to-decimal relationships. Although this study proved most encouraging, the approach has not been employed with more advanced mathematical concepts. Important to note is that although the experimental preparation employed a computer voice synthesizer to provide opening instruction to participants regarding response contingencies, neither the experimenter nor the voice synthesizer provided participants with any *rules* regarding the trained mathematical relationships. During training, selection of the correct comparisons simply resulted in a computer-generated verbalization of the word "yes." In discussing their findings, Lynch and Cuvo (1995) note that modification of their strategies might be needed and that teaching students to learn broad-spectrum rules during training might have enhanced the generalized math performance of their participants.

Lynch and Cuvo's (1995) protocol is consistent with the "traditional" experimental arrangements in equivalence and relational frame preparations (cf. Leader & Barnes-Holmes, 2001). To date, most stimulus equivalence and Relational Frame Theory (RFT) studies have required participants to perform a series of conditional discriminations during matching-to-sample (MTS) training. These include multiple stimulus relations such as greater than, less than, different from, opposite from, and equal to some arbitrary stimulus. Experimental preparations often involve a series of preliminary training trials with a nonarbitrary stimulus array to produce the contextual cues for establishing these relations (e.g., Smeets, Barnes-Holmes, Akpinar, & Barnes-Holmes, 2003; Steele & Hayes, 1991). Similar approaches have explored a wide variety of multiple stimulus relations. In describing a study of contextually controlled equivalence classes by Wulfert and Hayes (1988), Barnes-Holmes, Hayes, Dymond, and O'Hora (2001) suggest that "combining the different relational and functional contexts, one hundred and twenty untrained sequences among all of the stimuli emerged from only eight trained sequences for all subjects" (p. 64).

Nevertheless, researchers in the area of RFT as well as stimulus equivalence have been reluctant to invoke *rule-governance* to account for the establishment of multiple stimulus relations (Hayes, Barnes-Holmes, & Roche, 2003). In computerized versions of such experiments (e.g., Stewart, Barnes-Holmes, Roche, & Smeets, 2002), the sample stimuli may appear on one segment of the screen, and following a delay, comparison stimuli are displayed in various locations and configurations. Participants may select a comparison by clicking a mouse over a comparison stimulus and obtain accuracy feedback in the form of "Right" or "Wrong" posted on the screen. This is a particularly robust and reliable preparation for basic research in equivalence relations; however, as a strategy for teaching complex verbal relations, it is not very efficient. Of course, this type of experimental arrangement was never intended to function within a direct instruction protocol.

As a practical matter, it is virtually inconceivable that mathematics instructors (or designers of computer-assisted mathematics software) would be willing to have their students/software users attempt to learn complex mathematical relations without first providing them with some very precise rules for the relevant problem-solving behavior (Ninness, McCuller, & Ozenne, 2000). This does not preclude the adaptation of RFT procedures to applied instructional settings. It does, however, suggest a need for modification of the traditional computer-interactive platforms in building human computer-interactive applications that incorporate strategies based on RFT. In fact, many researchers have expressed the need for the alternative methodologies that include rules that apply to Relational Evaluation Procedures (REP) in order to generate more complex relational performances that are not possible using traditional MTS formats (D. Barnes-Holmes, personal communication, April 30, 2003). Moreover, REP would appear congenial with computer-assisted tutorial strategies aimed at training the *relationships among relationships* for many topics in applied and theoretical mathematics.

In our present development of computer-interactive math modules, we addressed the area of transformation of graphs of functions. Many mathematical functions have graphs that are related to one another in terms of their various types of vertical and horizontal shifts and transformations. For example, for a vertical transformation, the addition of a positive constant after a set of parentheses produces a shift upward (positive), while the subtraction of a constant causes a shift downward (negative). In the case of a horizontal transformation, the addition of a positive constant inside of a set of parentheses generates a shift to the left (negative), and the subtraction of a constant inside a set of parentheses produces a shift to the right (positive). Thus, horizontal transformations are somewhat "counterintuitive," and many students find it difficult to identify graphed representations of such transformations when they are arranged in multiple combinations of simultaneous horizontal and vertical shifts within a given formula. Nevertheless, understanding the mutually entailed relationships among sets of functions may allow students to derive more complex combinations of relations for a wide range of formulas and graphed analogues. New REP strategies may have practical applications for teaching such complex verbal relations.

As described by Barnes-Holmes et al. (2001), REP provides subjects with an opportunity to identify completely novel sets of stimulus relations that were not included in the training protocol. Unlike the bidirectional assessment of symmetry, REP strategies allow subjects to identify nonsymmetrical but related stimulus functions. Barnes et al. point out that, in experimental arrangements, words such as *If* and *Then* can influence the derived relations among words in a given sentence. The following is used as a naturalistic example of nonsymmetrical verbal relations: "If it rains, then take the car" (p. 70). They note that taking the car cannot cause the rain; however, raining will influence the likelihood of the car being taken. In an experimental arrangement, a given formula may generate a unique graph; however, the same graph may be described by an almost infinite array of formulas. Here the mathematical relationships that are reported on by subjects (formula-to-graph) are not specifically symmetrical. Showing subjects novel formulas allows them to identify a vast network of relationships that are extensions of the exemplars provided during training and pretraining.

In RFT, the expression *mutual entailment* denotes the essential bidirectionality of such relational responding, even when the correspondence is not specifically symmetrical. For example, if a particular mathematical value x is greater than y, then y is less than x. If x values vary inversely with y values, then y values vary inversely with x values and so on. As stated by Hayes, Fox, et al. (2001), mutual entailment "serves as a more generic term for what is called 'symmetry' in stimulus equivalence. Mutual entailment is a defining characteristic of arbitrarily applicable relational responding" (p. 29).

If we train mathematically naïve participants (pretested to rule out understanding of such concepts) to recognize particular features of math symbols and functions, we might demonstrate that they are able to develop a repertoire of relational responding that applies to similarly constructed formulas and graphed representations of these formulas. Moreover, the student who learns this relation as it pertains to one set of functions and who is told that many other functions operate in much the same manner may be able to apply the entailed relations to a much wider set of previously untrained functions.

Although the following study deals with learning the *transformation of mathematical functions*, it does not directly address transformation of functions as defined within RFT. Transformation of functions (from an arbitrarily applicable relational responding point-of-view) must show derived transfer of stimulus controls through equivalence relations using discriminative functions. As pointed out by Hayes, Barnes-Holmes, & Roche (2001), to demonstrate *transformation of function*, the stimulus functions themselves must be shown to be under contextual control. In accordance with these definitions, we will refer to the behavioral effects we address as *relating mathematical relationships* that are mutually entailed within frames of coordination (see Hayes, Barnes-Holmes, et al., 2001, for a complete discussion) where new learning includes a sequence of responses from an appendable network of related concepts in basic or advanced mathematics.

The following study focuses on the learned relationships that emerge following rather brief mathematical instructions in conjunction with MTS training. Specifically, the study addresses the complex behaviors that can come under the control of the mutually entailed features of mathematical formulas and their graphed analogues (Ninness et al., 2003). These relationships go beyond the formal properties of the particular relata employed during training, and they may be important because they provide basic repertoires from which more complex concepts are generated.

Experiment 1

Method

Participants and setting. Fifteen participants (7 male and 8 female) ranging from 18 to 35 years of age were recruited among university students as well as employees from a local hospital rehabilitation facility. Following informed consent and a pretest to determine level of familiarity with algebraic functions, individuals who demonstrated any familiarity with mathematical reflections or shifts were excluded from the study. Student participants earned 3 extra points on their first class examination plus $3.00 for taking part in the study. Because the hospital employees were not enrolled at the university, their reinforcement options were limited to financial reimbursement ($3.00) for their participation. The experimental sessions were conducted in unoccupied classrooms on the university campus or offices in the rehabilitation facility. The classroom and office environments were arranged to preclude interruptions, and they remained free of noise or other types of distractions.

Apparatus and software. The algebraic instructional and assessment software, written by Chris Ninness, was in Microsoft Visual Basic 6 for IBM PC compatible machines. Additionally, he developed a modified version of the self-organizing map (SOM) algorithm in C++ to classify user error patterns during experimental sessions, based on Kohonen's logic for the SOM algorithms (see Kohonen, 2001, for a general discussion of this architecture).

These two programs were integrated to provide a platform upon which all experimental arrangements were conducted and upon which all error patterns were classified. The experimental procedures were carried out on a Hewlett-Packard Pavilion ze5170 (Pentium [4] 2 GHz processor with 512 MB RAM) with an attached infrared mouse (or similarly configured IBM compatible laptop or desktop machine). The software provided math instructional tutorials and displayed graphs, and it also assessed and recorded the speed and accuracy of user performance during all phases of the study.

Design and procedure. The study was designed to simulate supplemental web-based tutorials for classroom mathematics instruction. Following informed consent, participants who demonstrated absolutely no familiarity with algebraic and trigonometric operations relative to vertical and horizontal shifts were asked to continue with the experiment. They received a brief (approximately 10 min) presentation, read aloud from note cards. The lecture included projections of formulas and graphs on an overhead screen above the lectern. All participants were exposed to the same introductory lecture and visual presentation regarding the relationship between particular formulas and their respective graph functions.

The lecture was immediately followed by didactic computer-assisted instruction. During this initial training sequence, participants did not choose correct comparisons. Per our attempt to parallel traditional classroom instruction supplemented by computer-assisted instruction, participants received computer-posted rules regarding the relationship between particular formulas and their graphs. Figure 1 illustrates one of the training screens addressing the relationship between formulas and graphs of these functions.

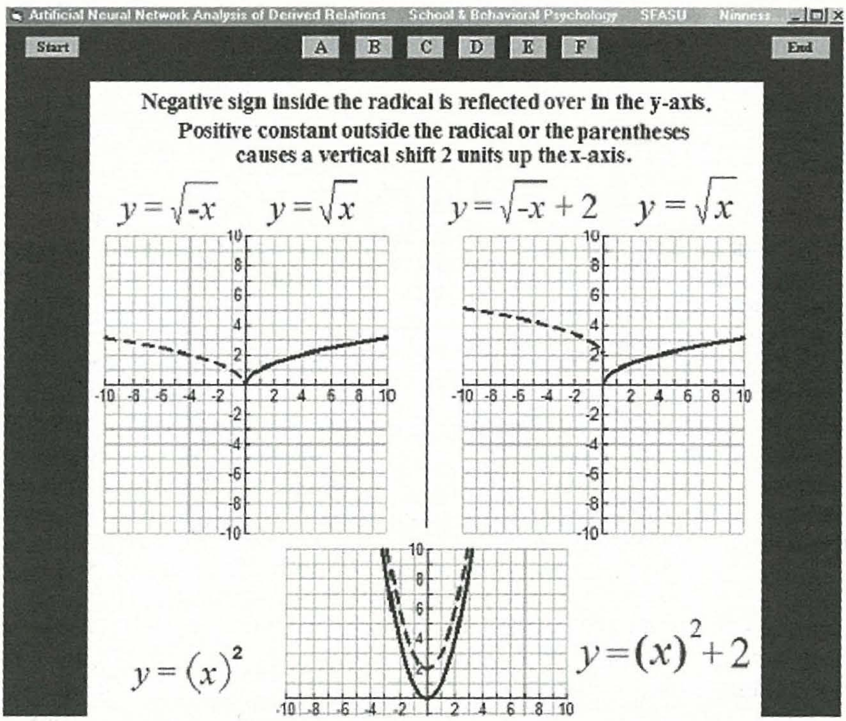

Figure 1. First training screen showing the reflection in the *y* axis and vertical shift when a negative sign is placed inside the radical and a constant is added outside of the radical. The square function has a vertical shift up the *x* axis when a constant is added outside of the parentheses.

On this training screen, participants were shown that the basic square root and square function are transformed when negative inputs and positive constants are provided. This was demonstrated graphically using solid (and blue on computer screen) lines to represent the basic square root and square function relative to dashed red lines to show various transformations of these mathematical functions.

Each training screen was supplemented by audio output from the computer that provided the same information as visually posted at the top of each screen. Thus, for each instructional screen, participants heard and read rules regarding vertical and horizontal shifts as they applied to the transformation of square and square root functions. To make these rules more salient and "memorable," participants read the rules aloud into a computer-housed microphone immediately after hearing them from the computer; however, the computer microphone served no actual purpose in this experiment beyond acting as a prop (perhaps prompt) for participants to overtly vocalize the rules displayed on the screen before them.

On the second instructional screen (Figure 2), participants could see (and hear) that, relative to the solid blue line, the basic square root and square function reflected down in the *x* axis when negative signs were positioned in front of the radical or parentheses. (Note: On several screens, we intentionally exaggerated the use of parentheses to emphasize precedence of particular operations.)

In the last training screen, shown in Figure 3, the square root and square function were shown to reflect in the *x* axis when negative signs

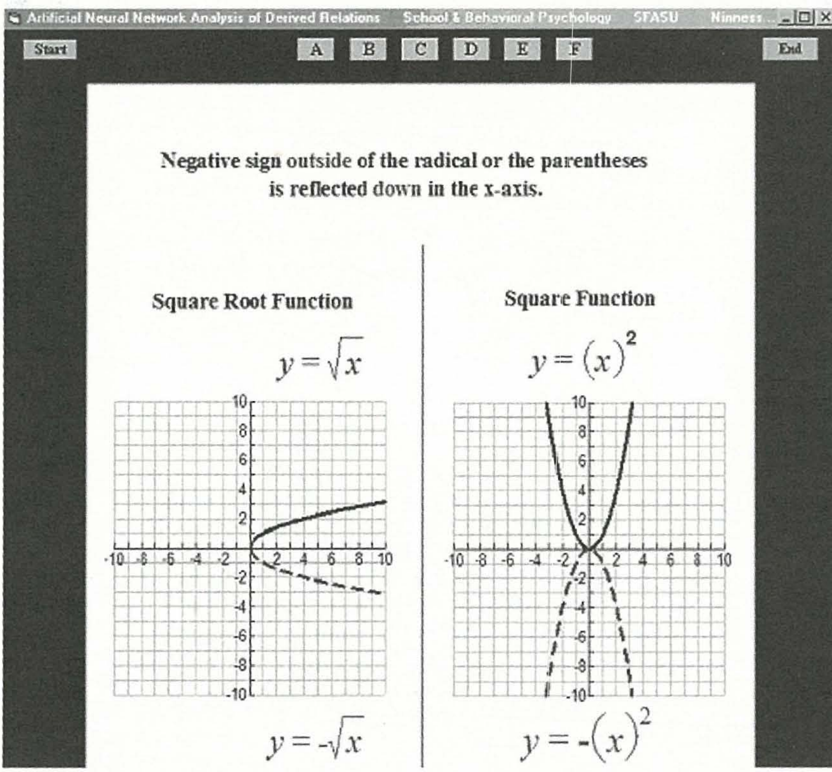

Figure 2. Second training screen showing changes in the square function when a negative sign is placed in front of the radical or parentheses.

preceded the primary functions and positive constants were added inside the function. It also showed that, relative to the solid blue line, adding a constant value inside the radical or the parentheses caused a horizontal shift *c* units in the opposite direction.

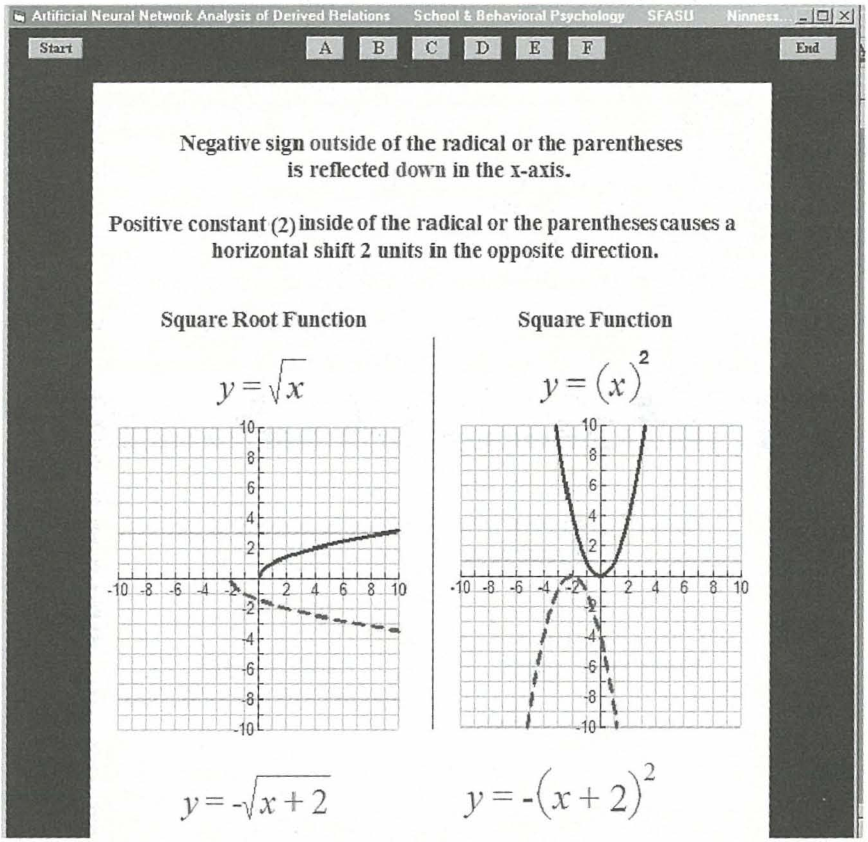

Figure 3. Third training screen showing the effect of a negative sign preceding the radical or parentheses in conjunction with a constant added inside of either function.

Participants were then tested for mutual entailment on formula-to-graph and graph-to-formula functions. If a particular relationship was not identified correctly during these tests for mutual entailment, the software took participants back to the initial training modules and retrained them on all of the original relationships. Subsequently, it provided the participants another opportunity to demonstrate formula-to-graph mutual entailment on the three square root functions that included a negative sign inside the radical, a negative sign before the radical, and a negative sign before the radical with a positive constant (2) inside the radical as shown below:

$$y = \sqrt{-x}, \quad y = -\sqrt{x}, \quad y = -\sqrt{x+2}.$$

(Tests for mutual entailment were conducted *only* on the *square root* functions that were employed during the above training screens.)

Figure 4 shows one example of a test for mutual entailment. Participants were exposed to conditional discrimination MTS tests for mutual entailment regarding the three previously instructed square root relations.

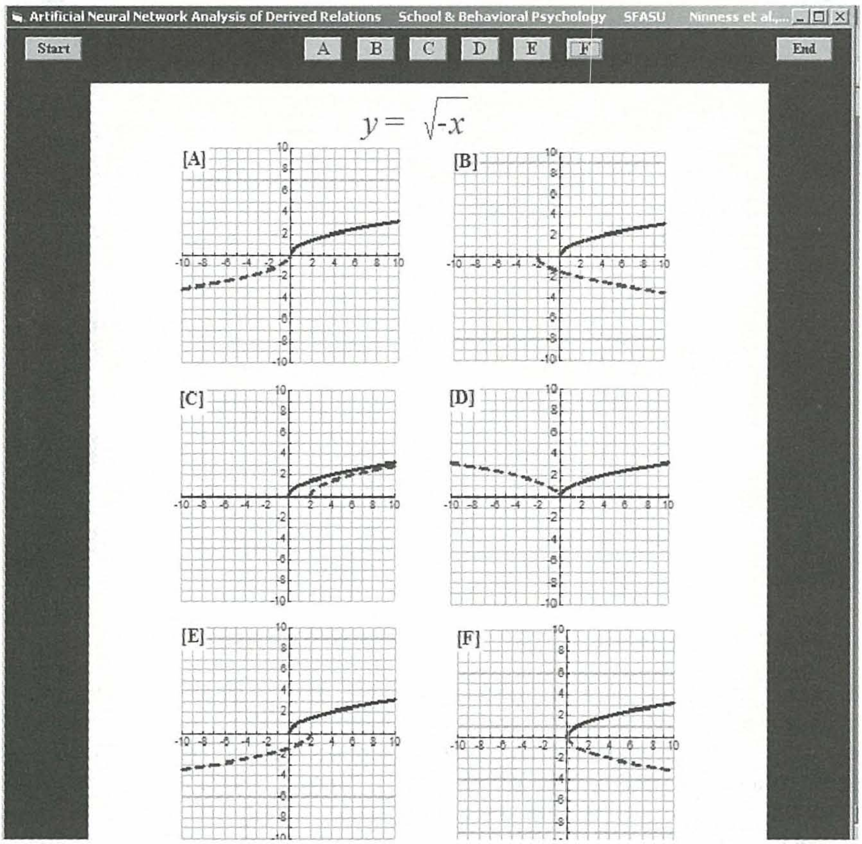

Figure 4. One of the tests for the property of mutual entailment following instructions regarding how graphs change according to formula details.

The tests for mutual entailment required that participants match formulas to graphs for all three of the above square root formulas on two consecutive trials. For example, correctly selecting [D] in the above screen moved participants to a test for mutual entailment, as shown in Figure 5, where [E] is the correct answer.

After participants demonstrated mutual entailment on two consecutive trials that included all three functions, they were assessed for 36 MTS novel relations on similarly constructed formulas and graphed analogues.

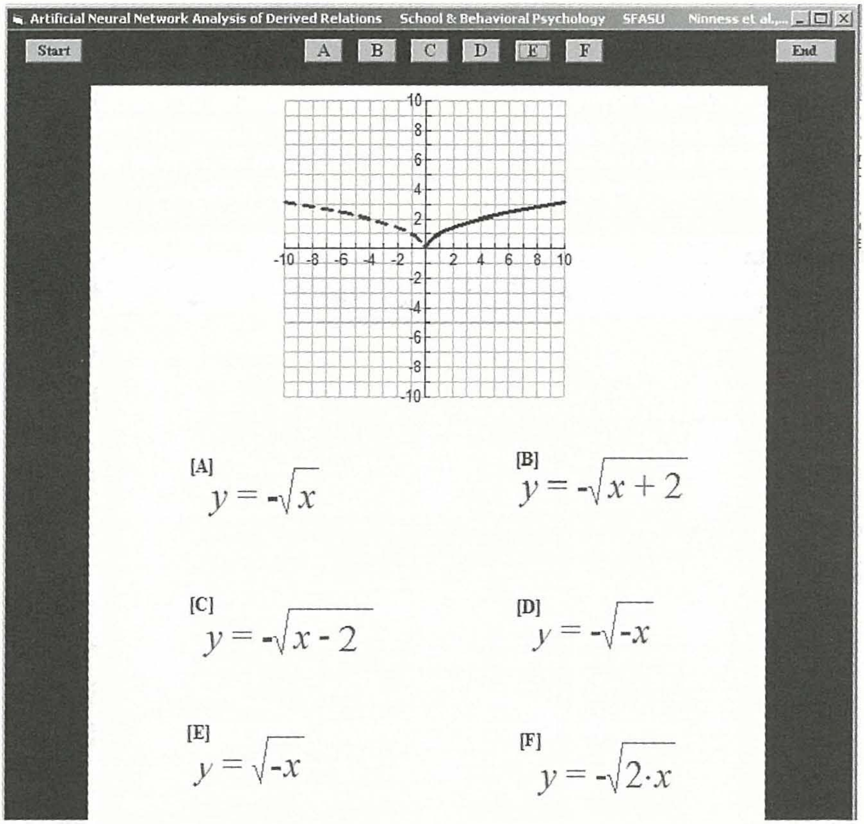

Figure 5. Test for mutual entailment following instructions regarding how graphs change according to formula details.

Accuracy feedback was not provided during any of these tests for novel relations. These 36 algebraic and trigonometric functions were variations of the originally instructed relationships described earlier. That is, novel test items required participants to identify a wide range of mathematical relations from an array of graphs that were not trained during any of the previous instructional arrangements. Moreover, these test items entailed identifying complex combinations of multiple constants with multiple combinations of positive and negative valences in the prefix and postfix operations.

Test items assessed participants' ability to identify graphed reflections and vertical and horizontal shifts for cubic, square, logarithmic, exponential, and sine formulas. Figures 6 and 7 show 2 of the 36 test screens that assessed novel relations. We hypothesized that learning the relations among mathematical relations (in this case formulas and graphs) for vertical and horizontal shifts together with those that describe reflections in the *x* axis and *y* axis might allow participants to identify new combinations of graphs.

In the Figure 6 example, participants who had successfully acquired

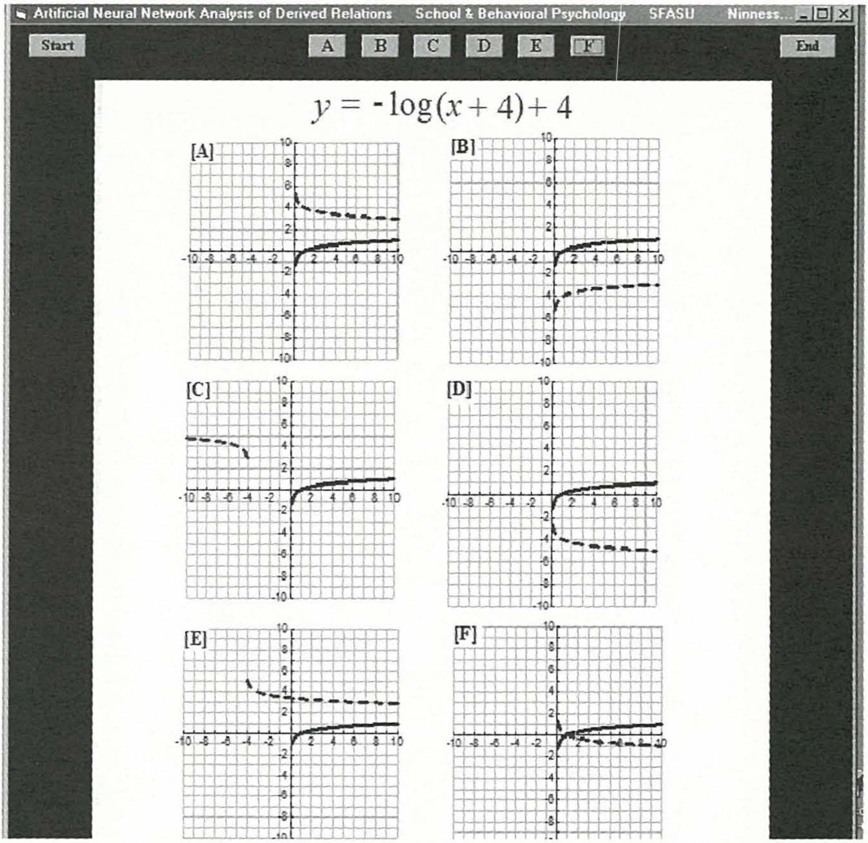

Figure 6. One of 36 screens that tested novel relations following computer-interactive instructions and MTS training.

the rules for negative prefix, positive constants within parentheses, and positive constants outside of parentheses might correctly select [E] as the combined effect when viewed as a graphed illustration of this function. In Figure 7, participants who had successfully acquired the rules for negative prefix and negative constants outside of parentheses might correctly select [A] as the combined effect of these variables and constants on a graphed illustration of this function.

The center of Figure 8 shows the four types of mathematical functions (four boxes within the center oval) described during training and those formulas (on the periphery) used as stimuli to test emergent relations as illustrated in Figures 6 and 7. (Note: A negative sign *inside* the parentheses of the *square function* reflects down in the same way as a negative sign outside of the parentheses. Thus, as indicated in the center of Figure 8, only the *square root* version of this formula was provided in training.)

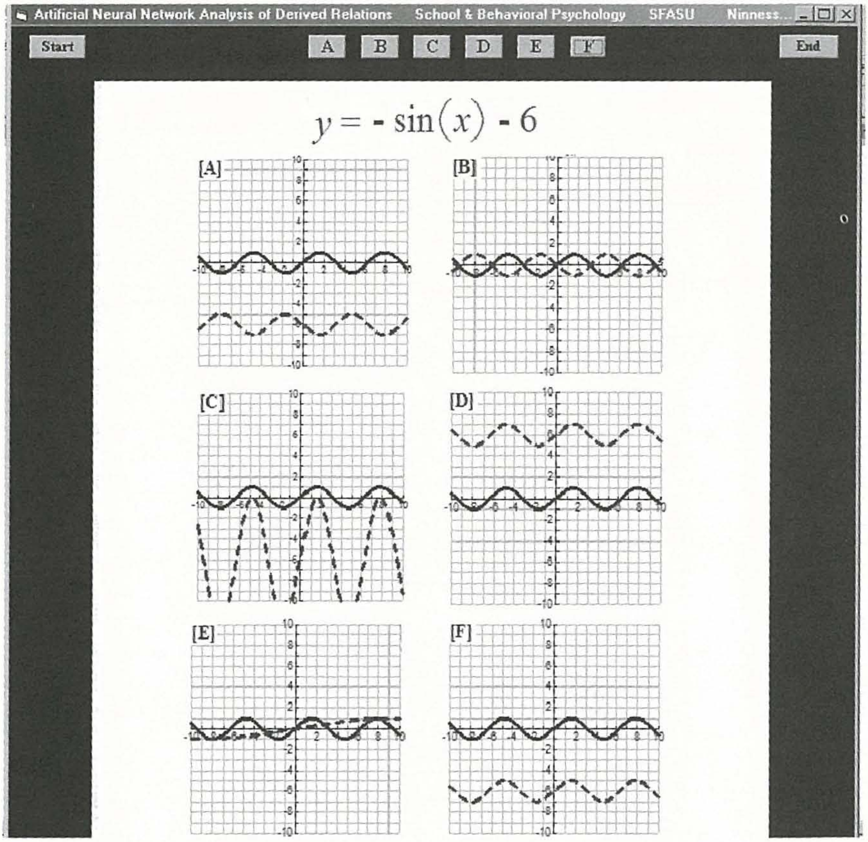

Figure 7. One of 36 screens that tested novel relations following computer-interactive instructions and MTS training.

Results

Of the 15 participants, 5 demonstrated perfect (or near perfect) performance on all of the above relationships between the formulas on the periphery and their respective graphic illustrations (3 of these 5 participants missed one item). Five other participants scored at or above 83% accuracy, and 2 scored above 60%. Only 3 participants scored below 60% during Experiment 1. Figure 9 shows the scatter of accurately derived relationships (blocks containing the digit 1) relative to errors (shaded blocks containing 0) prior to pattern analysis with the artificial neural network SOM (Kohonen, 2001).

There have been over 4,000 published variations and extensions of the original SOM (Kohonen, 2001), and it has proven to be especially useful in identification of hidden pattern formations, or data-clustering, within chaotic systems. Figure 9 illustrates that the pattern in which these errors occurred was not immediately discernable.

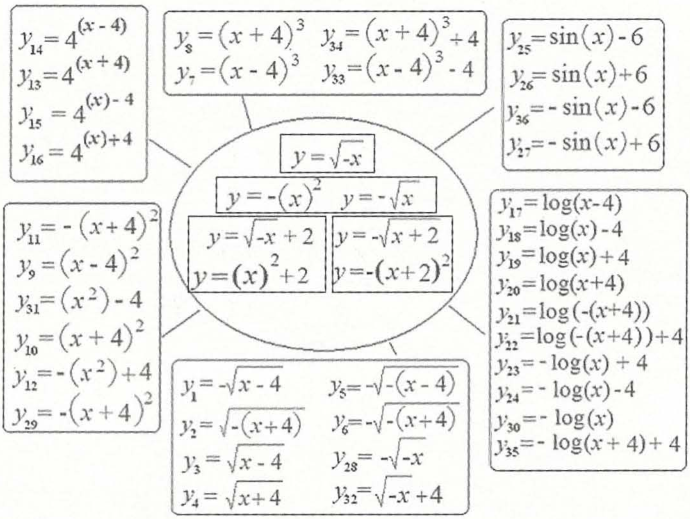

Figure 8. The center of this figure shows the four types of mathematical functions (four boxes within the center oval) described during training and those formulas (on the periphery) used as stimuli to test novel relations.

An adaptation of the SOM algorithm was developed by Chris Ninness as a diagnostic instrument for classifying error patterns that appeared during the first experiment. We employed the SOM to classify the error patterns across the 36 test items and 15 participants. Figure 10 shows the same data, but classified by error patterns with the SOM.

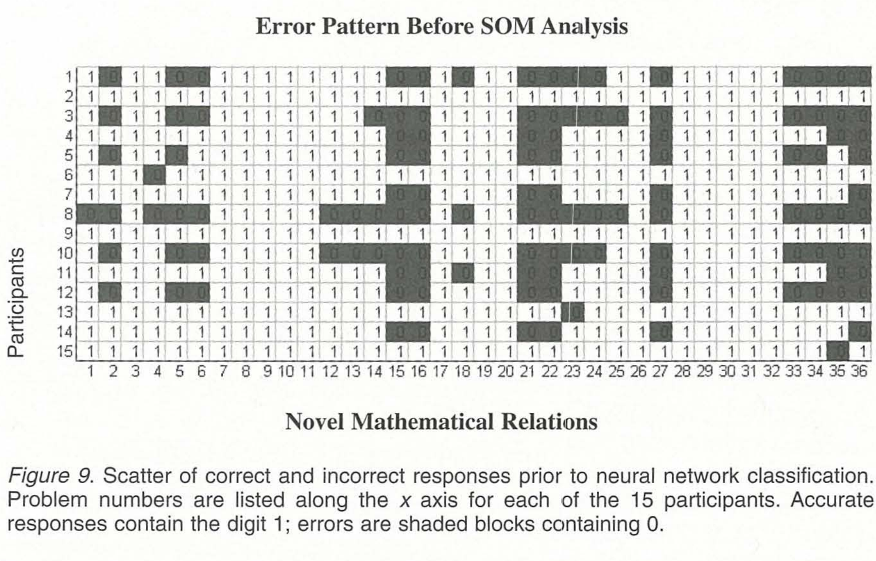

Figure 9. Scatter of correct and incorrect responses prior to neural network classification. Problem numbers are listed along the *x* axis for each of the 15 participants. Accurate responses contain the digit 1; errors are shaded blocks containing 0.

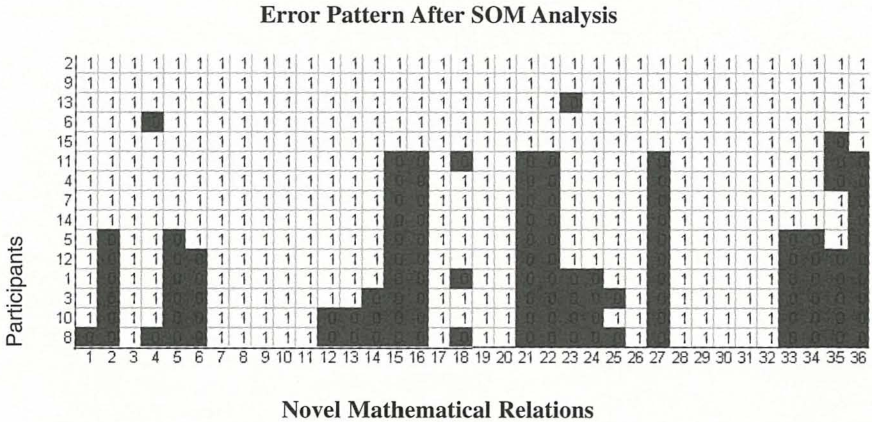

Figure 10. Classification of correct and incorrect responses following neural network classification. As in Figure 9 above, problem numbers are listed along the *x* axis for each of the 15 participants. Accurate responses contain the digit 1; errors are shaded blocks containing 0.

Experiment 2

Consistent with simulating a natural context, in which students receive primary instruction in the classroom and obtain supplemental material online, Experiment 2 was designed to remediate the errors that occurred among participants during Experiment 1. Classification of error patterns allowed us to identify difficult items across participants for various types of mathematical relationships. Several of the formulas for which these participants failed to derive graphed representations during Experiment 1 are shown below.

$$y = -\sqrt{-(x+4)} \quad y = -\log(x+4) + 4 \quad y, = \log(-(x+4)) + 4 \quad y = -\sin(x) - 6$$

Although these formulas obviously represent very different mathematical functions, they show common behavioral functions. Identifying the graphs of these formulas required relating new forms of untaught sequence responses from a fairly detailed relational network of algebraic and trigonometric rules and their associated graphs on the coordinate axis. In the above formulas, three or more mathematical stimuli (signs, symbols, constants) interact within a specific context. For example, a negative sign in front of the radical (or log or sine function) causes the functions to shift downward, while a negative sign inside the radical (or log function) represents a reflection over in the *y* axis. Within the same examples, a constant inside the parentheses (i.e., 4) moves the function in the opposite direction along the *x* axis. Moreover, a negative or positive constant (i.e., 4 or 6) placed outside of the radical or the parentheses causes the function to shift up or down the *y* axis. This contextual relation (C_{rel}) applies to the formulas and graphs of these functions within the very limited relational network we assessed in this study.

With the exception of the two exponential functions, the error pattern displayed by the SOM revealed that most participants were no more likely to miss novel functions (e.g., sine, cube, or log) than they were to miss square or square root functions when these functions were displayed in their most rudimentary configurations. Accordingly, training in Experiment 2 emphasized changes in the square root functions relating to the effects of combinations of negative and positive signs in front of and inside the radicals in conjunction with positive and negative constants inside radical signs.

Method

Participants and setting. Of the participants in Experiment 1, 10 served in Experiment 2. Participants 2, 9, 13, 6, and 15 were not included in Experiment 2, as they demonstrated perfect, or near perfect, acquisition of derived relationships during the first experiment. The remaining participants (11, 4, 7, 14, 5, 12, 1, 3, 10, and 8) were invited to participate in a brief remediation session in which verbal instructions and computer-interactive tutorials were used to provide additional training in light of the types of errors classified by the SOM. This second experiment was conducted under the same conditions provided in Experiment 1.

Design and procedure. A brief lecture was read from note cards, and all participants received the same rules for reflections and shifts in the square root functions relating to the effects of combinations of negative and positive signs in front of and inside the radicals in conjunction with positive and negative constants inside radical signs. As in the original instructional sequence in Experiment 1, the effects of constants positioned after the radical sign (or parentheses) were briefly reiterated. No other functions were discussed or trained in Experiment 2.

Following the oral lecture, participants were reassessed for accuracy of problem solving according to the same MTS protocol implemented in Experiment 1. Thus, training and tests for mutual entailment only addressed more complex square root functions and a review of the original material provided in Experiment 1. As in Experiment 1, participants demonstrated mutual entailment between all formulas and their respective graph analogues before the computer allowed them to be reassessed for novel relations identified in Figure 8.

Results

Figure 11 shows reduction in participant errors following targeted treatments based on the SOM classification of errors. All participants demonstrated a reduction in errors relative to their performance in Experiment 1.

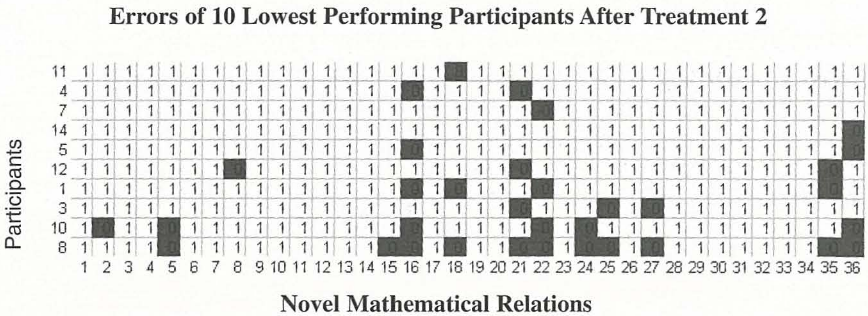

Figure 11. Pattern of errors following interventions based on errors identified by the SOM. Incorrect items are shaded and contain 0s.

Discussion

Brief instruction and MTS training aimed at a small set of formulas to graphs and graphs to formulas enabled participants to derive the relations for a series of complex combinations of formulas and their associated graphs. Novel relations entailed new multiple combinations of constants within parentheses and radicals with combinations of positive and negative signs in the prefix and postfix operations for more intricate algebraic and trigonometric functions. These results are consistent with RFT and stimulus equivalence studies demonstrating that training a small number of basic relations may generate the emergence of many more detailed relations beyond the specific trained relations. Even though this study did not test for the combinatorial mutual entailment or equivalence-equivalence relations, the evidence suggests that learning a small number of relations and testing for the property of mutual entailment may be sufficient for generating enhanced relational networks (Stewart, Barnes-Holmes, Roche, & Smeets, 2001).

Because all of the mathematical formulas employed during this study have graphs that are transformations of the more basic graphs employed during training, it was possible for participants to derive the mutually entailed relations for complex variations of square, square root, exponential, cubic, and logarithmic functions. That is, for a vertical transformation, the addition of a positive number inside the parentheses or radical generates a shift up in the positive direction and subtraction of a constant moves the function down in the negative direction. For a horizontal transformation, the addition of a positive constant inside the parentheses (or radical sign) produces a shift in the negative direction, and the subtraction of a constant generates a shift in the positive direction. It is hard to imagine that a participant could make any of the above relevant discriminations with regard to new formulas and their respective patterns on the coordinate axes without employing specific mathematical rules. Indeed, the debriefing of all participants suggests that those who were able to clearly express the primary rules supplied during training were more likely to perform efficiently. And, it appears that the ability to express these rules at least facilitated the development of the complex relational networks needed to perform these operations. By the end of Experiment 2, all participants were able to paraphrase the correct rules regarding the formula's influence on vertical and horizontal shifts and reflections in the *x* and *y* axes. However, knowing these rules may not be sufficient to operate with them accurately. None of our pilot participants who were exposed to the same instructions were able to respond correctly to new formula-to-graph relations until they were able to consistently demonstrate the property of mutual entailment on similar types of functions.

The elegance in many mathematical principles flows from the highly specific relational functions that are required; however, mathematical operations entail very few nonrelational functions. In this study, training participants to respond accurately to a relatively small number of stimulus sequences generated new combinations of more complex mathematical progressions. Our experimental preparations involved the relating of derived relational networks in the sense that participants had to add or remove particular elements such as negative or positive signs when particular types of formulas were presented on the computer screen. Thus, the *transformation of mathematical functions* in this study addressed C_{rel} control and the relevant entailment process. It is not clear, however, that any of our procedures addressed contextual function (C_{func}) control or the transformation of nonrelational functions (D. Barnes-Holmes, personal communication, April 30, 2003).

It is possible that the immediate performance improvement of Participants 2, 9, 13, 6, and 15 may have been caused by some advanced training history prior to our investigation; however, discussion with these participants and pretesting suggest that they were unfamiliar with these concepts and were unable to demonstrate these similar types of discriminations prior to treatment. As a caveat, these relations were tested in only one direction. Subsequent research should assess derived relations in both directions. Nevertheless, all 15 participants were pretested, and none of the participants were able to successfully identify any of the algebraic formulas relative to their graphic representations. It is also possible that a practice effect may have beneficially influenced participants' performance during Experiment 2; however, pilot testing and retesting on the same math items has consistently failed to show any noticeable gains in subject performance unless specific rules regarding mathematical relationships are provided and MTS procedures are implemented.

Our findings suggest that error patterns displayed by students are not caused by carelessness alone, nor are they exclusively a function of insufficient practice. Many students make incorrect inferences during instruction. Indeed, we have found that some incorrect problem-solving strategies may produce correct answers intermittently (See Ninness & Ninness, 1998, 1999, for a discussion). Under such conditions, many students become increasingly frustrated and confused. Our particular neural net adaptation of the Kohonen SOM operates somewhat analogously to a descriptive analysis in that it identifies the conditions (types of problems) that are associated with errors and accurate mathematical discriminations. In doing so, it provides feedback to the software developer as to where more intensified and detailed tutorials are required. Of 15 participants, 10 needed supplemental training to derive all (or nearly all) relations successfully. Having access to a pattern of conceptual errors allowed us to intervene more efficiently. With the SOM, we were able to provide brief but specific remediations regarding interconnected items that our participants found particularly challenging.

Future preparations might address *combinatorial mutually entailed mathematical relationships* that emerge from computer-posted instructions and enhanced MTS procedures. Our lab is focusing on the *combinatorial mutually entailed* properties of the *standard* forms of many complex formulas, the *factored* forms of these same formulas, and their mutual graph analogues. Similar investigations might survey the combination of new REP strategies in conjunction with artificial neural networks to design more effective web-based instructional platforms for training math skills. Rather than depending on student *discovery* of fundamental mathematical relationships, we should be able to provide online materials using new types of REPs to more efficiently train and assess student mastery of the primary relationship among a multitude of critical mathematical relationships. In circumstances where we have not provided adequate computer-interactive training regarding these relationships, the SOM might be able to provide feedback as to where the relationship among relationships is not emerging as anticipated and where better training protocols are needed. Such systems could allow students to develop the critical hierarchy of skills to more efficiently negotiate the space between graphs and formulas.

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